Recursive Algorithm for the Reliability of Connected-(1,2)-or-(2,1)-out-of- (m,n): Linear and Circular Lattice Systems

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Abstract

The connected (r,s)-(s,r)-out-of-(m,n): F lattice systems fail if and only if at least any connected subset of (r,s) or (s,r) of failed components occurs. For example the connected-(1,2)-or-(2,1)-out-of-(m,n): F lattice systems fail if at least a (1,2)-matrix (i.e. a row with 2 components) or a (2,1)-matrix (a column with two components) of failed components occurs. Many researchers set numerous algorithms to compute the reliability of such systems. In this paper, a new recursive algorithm for evaluating the reliability of a connected-(1,2)-or-(2,1)-out-of-(m,n): F linear and circular system is obtained. We calculate the orders of the computing time of the proposed algorithm, in order to compare it with the algorithms in the previous studies, the comparison shows that the proposed algorithm is more efficient than the other algorithms in the previous studies.

Keywords: Consecutive k-out-of-n: F system, Connected-X-out-of-(m,n):F system.

Notations:

\( I_{n}^{1} \) \hspace{1cm} The indices of the components numerated from 1 to \( n \) i.e. \( \{1,2,\ldots,n\} \)

\( P(I_{n}^{1}) \) \hspace{1cm} The power set of \( I_{n}^{1} \).

\( p_{j}(q_{j}) \) \hspace{1cm} The reliability (unreliability) of the \( j^{th} \) component, and

\[
p_{w} = \prod_{j \in W} p_{j} \cdot q_{w} = \prod_{j \in W} q_{j} \forall W \subseteq I_{n}^{1}\]

\( p_{j}^{i}(q_{j}^{i}) \) \hspace{1cm} The reliability (unreliability) of the \( j^{th} \) component at the \( i^{th} \) layer (circle).

\( p_{w}^{i}(q_{w}^{i}) \) \hspace{1cm} The reliability (unreliability) of the \( i^{th} \) layer (circle), when the indices failed components labeled by the set \( W \),

\[
p_{w}^{i} = \prod_{j \in W} p_{j}^{i} \cdot q_{w}^{i} = \prod_{j \in W} q_{j}^{i} \wedge W \subseteq I_{n}^{1}\]

\( R^{i}(W) \) \hspace{1cm} \( P\{Z_{i} = W\} = p_{w}^{i} q_{w}^{i} \)

\( R_{L(C)}(i) \) \hspace{1cm} The reliability of the connected-(1,2)-or-(2,1)-out-of-(i,n): F linear (circular) subsystem

\( R_{L(C)}(i,X) \) \hspace{1cm} The reliability of the connected-(1,2)-or-(2,1)-out-of-(i,n): F linear (circular) subsystem when \( X \) represents the failed component in the \( i^{th} \) layer (circle) where \( X \subseteq I_{n}^{1} \).

1. Introduction

Kontoleon(1980) [5] was the first which studied the consecutive k-out-of-n: F system under the name ““r-successive-out-of-n: F system””. Later abundant and growing interest in studying the reliability of consecutive k-out-of-n: F systems due to its importance in applications (e.g. Telecommunication systems with \( n \) relay stations, the pipeline of transmit oil system, etc.), many generalizations are achieved in ([1], [2], [6], [7] and [9]), the system consists of \( n \) components and fails if at least \( k \) consecutive components fail, it classified according to the connection between components into two types: linear and circular.

Salvia & Lasher [7] introduced the two dimensional consecutive k-out-of-n: F system, while Boehme et al. [1] introduced more general definition for the two
dimensional consecutive “the connected-X-out-of-(m,n): F linear and circular system”, it consists of (mn) components, where the linear system is arranged as a matrix with m rows and n columns, and the circular system arranged as m circles and n rays, (the intersections of circles and rays represent the elements). The system fails if the connected components represented by the set \( X \) fails, \( (X \) may be \( (r,s) \) or \( X = (r,s) - (s,r) \) \( s, r \leq m, n \) ), e.g. the connected-(1,2)-or-(2,1)-out-of-(m,n): F linear and circular system, the system fails if any 2 connected components fail at any row or column, [1]introduced a practical example for such systems, “the supervision system” when \( (m,n) = (4,4) \) as shown in the following figure.

![Figure 1: An out of observation area in the (1,2)-or-(2,1)-out-of-(4,4): F linear system](image)

Each TV camera can supervise a disk of radius \( r \), and the cameras in each row and column are of the same type and are a distance \( r \) from each other. The supervision system is failed if an area inside of the sketched square with sides \( 3r \) is out of observation. The system fails if (at least) two connected cameras (connected by a line) in a row or column fail. Failed elements are represented by black cameras. The black area between the cameras at the positions rows and columns respectively (2,2) and (2,3) is out of observation.

Further, investigations regarding the reliability of the connected-(r,s)-out-of-(m,n): F linear systems are given by Malinowski and Preuss [8], Yamamoto and Myakawa [10], and Yamamoto and Akiba[12]-[14]. In 2008, Yamamoto et al. [14] achieved recursive algorithm for evaluating the reliability of a connected-(1,2)-or-(2,1)-out-of-(m,n): F linear lattice systems only.

In this paper, the reliability of the connected-(1,2)-or-(2,1)-out-of-(m,n): F linear and circular lattice systems is obtained using a new recursive algorithm, it depends on representing the functioning states of the one dimensional consecutive 2-out-of-n: F linear and circular system, finally, we evaluate the orders of the computing time, and show that the proposed algorithm outperforms the algorithms presented in the previous studies.

The following assumptions are assumed to be satisfied by the connected-(1,2)-or-(2,1)-out-of-(m,n): F linear and circular lattice systems.

1. The state of the component and the system is either “functioning” or “failed”
2. All the components are mutually statistically independent.

2. The Consecutive 2-out-of-n: F Systems

Consider the one-dimensional consecutive k-out-of-n: F linear and circular system, and \( I^1_n = \{1, 2, ..., n\} \) denotes the possible labels (indices) of the failed components. We shall refer the system using the indices of the failed components, if the system is in the functioning state; we name the set of failed component by the functioning subset, otherwise the failed subset. (For example the consecutive 2-out-of-5: linear and circular system, the set \( X = \{1, 3\} \subseteq I^1_n \) or for simply 13, indicates that the 1st and the 3rd
components are failed, in spite of these failed components but the system still functioning, so the set \( X=13 \) is a functioning subset. On the contrary of the set \( X=12 \) is a failed subset. According to basics of probability, the failure space of the components of the consecutive of the 2-out-of-\( n \): F linear (circular) system \( P\left(1^1_n\right) \), we can divide it into two sub collection, \( \Theta^L(c) \) and \( \Psi^L(c) \) the functioning and failure space of the consecutive 2-out-of-\( n \): F respectively, \( \Theta^L(c) \) may be defined as the set of all functioning subsets \( X \) of \( 1^1_n \) for the linear (circular) system as:

\[
\Theta^L(c) = \{ X \in P\left(1^1_n\right) : \{r, r+1\} \not\subset X \} \text{ where } r \in 1^n_{n+1}, (r \in 1^n_1, \& n + 1 \equiv 1 \text{ for the circular system})
\]

3. The proposed algorithm

In this section, we propose a new recursive algorithm for the reliability of the connected-(1,2)-or-(2,1)-out-of-(\( m,n \)): F linear and circular system. First, we provide some logical relations and definitions that help us to formulate the main theorems which include the basic idea of the proposed algorithm.

Consider the connected-(1,2)-or-(2,1)-out-of-(\( m,n \)): F linear (circle) system, and let \( 1^1_n = \{1, 2, ..., n\} \) be the label of the components at any layer (circle). So the failure space of the components in any layer (circle) is \( P\left(1^1_n\right) \).

Assume that \( X \in P\left(1^1_n\right) \) represents the indices of the failed components of any layer (circle) in the connected-(1,2)-or-(2,1)-out-of-(\( m,n \)): F linear (circle) system, if the system in the functioning state, then \( X \) is a functioning subsets of the consecutive 2-out-of-\( n \): F linear (circle) system i.e. \( X \in \Theta^L(c) \), otherwise we have at least two connected failed components, which implies that the whole system fails (see figure2).

Define the random variable \( Z_i \) on the \( P\left(1^1_n\right) \) such that \( \{Z_i = X\} \) the event that the \( i^{th} \) layer (circle) has the indices of failed components \( X \in P\left(1^1_n\right) \) and \( A_i \left( X \right) \) the events that the system is in the functioning state where the set \( X \) represents the \( i^{th} \) layer (circle), this means that the \( \left(i-1\right)^{th} \) layer (circle) may be any subset \( Y \in \Theta^L(c) \) that guarantee that there is no common failed components i.e. \( Y \) must not intersect the failed components of \( X \in \Theta^L(c) \) in the \( i^{th} \) layer (circle), otherwise the whole system fails (see figure 2 &3), let \( \Theta^L(c) \left( X \right) \) be the conditional functioning space of the set \( X \).

\[
\Theta^L(c) \left( X \right) = \{ Y \in \Theta^L(c) : Y \cap X = \emptyset \}, \text{ note that } \Theta^L(c) \left( \emptyset \right) = \Theta^L(c), \text{ also let also } A_i \left( X \right) =
\]

\[
\begin{align*}
\{Z_i = X\} \cup \bigcup_{Y \in \Theta^L(c) \left( X \right)} A_{i-1} \left( Y \right) : i = 2, 3, ..., m
\end{align*}
\]

Theorem 2.1: for \( j=1, 2, ..., m \)

\[
R_{L(c)} \left( i, X \right) =
\]

\[
\begin{align*}
R^i \left( X \right) \sum_{Z \in \Theta^L(c) \left( X \right)} R_{L(c)} \left( i-1, Z \right) \quad & i \geq 2, X \in \Theta^L(c) \\
R^i \left( X \right) \quad & i = 1, X \in \Theta^L(c) \\
0 \quad & X \in \Psi^L(c)
\end{align*}
\]

then the reliability of the connected-(1,2)-or-(2,1)-out-of-(\( m,n \)): F linear (circular) system is \( R_{L(c)} \left( m, X \right) = \sum_{X \in \Theta^L(c)} R_{L(c)} \left( m, X \right) \)

Proof:

\[
R_{L(c)} \left( 1, X \right) = P \left\{ A_1 \left( X \right) \right\} = P \left\{ Z_1 = X \right\}
\]

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For $i = 2, 3, \ldots, m$
\[
R_{L(C)}(i, X) = P\{A_i(X)\} = \\
= P\left\{\bigcup_{Y \in \Theta^{R(C)}(X)} \left(\bigcap_{i \in Y} A_i(Y)\right)\right\} \\
= \sum_{Y \in \Theta^{R(C)}(X)} P\left\{Z_i = X \cap \left(\bigcap_{i \in Y} A_i(Y)\right)\right\} \\
= R^i(X) \sum_{Y \in \Theta^{R(C)}(X)} R_{L(C)}(i - 1, Y) \\
\]

And
\[
R_{L(C)}(i) = P\left\{\bigcup_{X \in \Theta^{R(C)}} A_i(X)\right\} = \sum_{X \in \Theta^{R(C)}} P\{A_i(X)\} = \sum_{X \in \Theta^{R(C)}} R_{L(C)}(i, X) \\
\]

4. Evaluation of the proposed algorithm.

Yamamoto et al. [14] introduced an $O\left(n^2 2^m\right)$ algorithm for the connected-(1,2)-or-(2,1)-out-of-(m,n): F linear system only, it is more efficient than Higashiyama[3] and Yamamoto [11] algorithms, which requires $O\left(2^{mn}\right)$ and $O\left(n m^2 2^m\right)$ computing times respectively.

Our proposed algorithm looks like the Yamamoto et al. Algorithm [14], where Yamamoto used the columns to find a reliability of the connected-(1,2)-or-(2,1)-out-of-(m,n): F linear system only, while we used the rows to find the reliability not only for the linear, but also for the circular system.
Since the number of the subsets in \( \Theta^t(C) \) in any row is less than \( 2^n \) and the number of \( R^t(X) \) is also less than \( 2^n \), which applies for \( i=1,2,\ldots,m \), so the order of computing time of \( R_{L(C)}(m) \) using our algorithm is \( O\left(m 2^{2n}\right) \). It is clear that the order of computing of our algorithm is less than or equal the Yamamoto et al.[14] algorithm when \( n \leq m \). (For the linear system if \( n > m \), we can rotate the system 90° degree to have the same computing time of Yamamoto et al.[14]).

Table 1 showsthat the proposed algorithm is very efficient and shorter than the other previous algorithms, clearly.

Table 1: Comparison in Computation Times

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5. Conclusion

In this paper, we proposed a new recursive algorithm for the reliability of a connected-(1,2)-or-(2,1)-out-of-\((m,n)\):\( F \) linear and circular lattice system. We evaluated the orders of the computing time. The proposed algorithm is more efficient than the algorithms in the previous studies.

Appendix

Example 1: Compute the reliability of connected-(1,2)-or-(2,1)-out-of-(3,2): \( F \) linear system when the components are independent identically distributed \( i.i.d. \).
\[ R_{L}(2) = \sum_{X \in \Theta^c} R_{L}(2, X) = \left[p^4 + 2p^3 q + \left[p^q + p^2 q^2\right] + \left[p^q + p^2 q^2\right]\right] = p^4 + 4p^3 q + 2p^2 q^2 \]

\[ R_{L}(3) = R^1(\varnothing) \sum_{Y \in \Theta^c(2)} R_{L}(2, Y) = p^2 \left[R_{L}(2, \varnothing) + R_{L}(2, 1) + R_{L}(2, 2) \right] = p^2 \left[\left[p^4 + 2p^3 q\right] + \left[p^q + p^2 q^2\right]\right] = p^5 q + 3p^4 q^2 + 3p^3 q^3 \]

\[ R_{L}(3, 1) = R^3(1) \sum_{Y \in \Theta^c(1)} R_{L}(2, Y) = p^4 \left[R_{L}(2, \varnothing) + R_{L}(2, 1) \right] = R_{L}(3, 1) = p^5 q + 3p^4 q^2 + 3p^3 q^3 \]

\[ R_{L}(3) = \sum_{X \in \Theta^c} R_{L}(3, X) = p^6 + 4p^5 q + 2p^4 q^2 \]

\[ = p^6 + 6p^5 q + 10p^4 q^2 + 2p^3 q^3 \]

**Example 2:** Compute the reliability of the connected-(1,2)-or-(2,1)-out-of-(3,4): F circular system when the components are i.i.d.

\[ \Theta^c = \{\varnothing, 1, 2, 3, 4, 13, 24\}, \Theta^c(\varnothing) = \Theta, \]

\[ \Theta^c(1) = \{\varnothing, 2, 3, 4, 24\}, \Theta^c(2) = \{\varnothing, 1, 3, 4, 13\} \]

\[ \Theta^c(3) = \{\varnothing, 1, 2, 4, 24\}, \Theta^c(4) = \{\varnothing, 1, 2, 3, 13\} \]

\[ \Theta^c(13) = \{\varnothing, 2, 4, 24\}, \Theta(24)^c = \{\varnothing, 1, 3, 13\} \]

\[ R^1(\varnothing) = p^4, R^1(1) = R^1(2) = R^1(3) = p^3 q, \]

\[ R^1(4) = p^2 q, R^1(13) = R^1(24) = p^2 q^2 \]

\[ R_{C}(2, 0) = R^2(\varnothing) \sum_{Z \in \Theta^c(2)} R_{C}(1, Z) = p^4 \left[p^4 + 2p^3 q + 2p^2 q^2\right] = p^8 + 4p^7 q + 2p^6 q^2 \]

\[ R_{C}(2, 1) = R_{C}(2, 2) = R_{C}(2, 3) = R_{C}(2, 4) \]

\[ R_{C}(2, 13) = R_{C}(2, 24) = R^2(13) \sum_{Z \in \Theta^c(13)} R_{C}(1, Z) \]

\[ = R^2(1) \sum_{Z \in \Theta^c(1)} R_{C}(1, Z) \]

\[ = p^3 q \left[p^4 + 3p^3 q + p^2 q^2\right] = p^7 q + 3p^6 q^2 + p^5 q^3 \]

\[ R_{C}(2, 13) = R_{C}(2, 24) = R^2(13) \sum_{Z \in \Theta^c(13)} R_{C}(1, Z) \]

\[ = p^2 q^2 \left[p^4 + 2p^3 q + p^2 q^2\right] = p^6 q^2 + 2p^5 q^3 + p^4 q^4 \]

\[ R_{C}(2) = \sum_{Z \in \Theta^c} R_{C}(2, Z) \]

\[ = R_{C}(2, \varnothing) + R_{C}(2, 1) + 2R_{C}(2, 13) \]

\[ = \left[p^8 + 4p^7 q + 2p^6 q^2\right] + 4 \left[p^7 q + 3p^6 q^2 + p^5 q^3\right] + 2p^5 q^4 \]

\[ = p^{12} + 8p^{11} q + 16p^{10} q^2 + 8p^9 q^3 + 2p^8 q^4 \]

\[ R_{C}(3) = R_{C}(3, 1) = R_{C}(3, 2) = R_{C}(3, 3) = R_{C}(3, 4) \]

\[ = R^3(1) \sum_{Z \in \Theta^c(1)} R_{C}(1, Z) \]

\[ = p^3 q \left[R_{C}(2, \varnothing) + 3R_{C}(2, 2) + R_{C}(2, 24)\right] \]

\[ = p^3 q \left[p^8 + 2p^7 q + 2p^6 q^2\right] \]

\[ = p^3 q \left[p^7 q + 3p^6 q^2 + p^5 q^3\right] \]

\[ + \left[p^6 q^2 + 2p^5 q^3 + p^4 q^4\right] \]

\[ = p^{11} q + 7p^{10} q^2 + 12p^9 q^3 + 5p^8 q^4 + p^7 q^5 \]

\[ R_{C}(2, 13) = R_{C}(2, 24) = R^2(13) \sum_{Z \in \Theta^c(13)} R_{C}(1, Z) \]

\[ = p^2 q^2 \left[R_{C}(2, \varnothing) + 2R_{C}(2, 2) + R_{C}(2, 24)\right] \]

\[ = R_{C}(2, 13) = p^2 q^2 \left[p^8 + 4p^7 q + 2p^6 q^2\right] \]

\[ = R_{C}(2, 13) = p^2 q^2 \left[p^7 q + 3p^6 q^2 + p^5 q^3\right] \]

\[ + \left[p^6 q^2 + 2p^5 q^3 + p^4 q^4\right] \]

\[ = p^{10} q^2 + 6p^9 q^3 + 7p^8 q^4 + 5p^7 q^5 + p^6 q^6 \]
\[ R_c(3) = \sum_{X \in \mathcal{F}} R_c(3, X) \]
\[ = R_c(3, \emptyset) + 4R_c(3, 1) + 2R_c(3, 13) \]
\[ = [p^{12} + 8p^{11}q + 16p^{10}q^2 + 8p^9q^3 + 2p^8q^4] \]
\[ + [4p^{11}q + 28p^{10}q^2 + 48p^9q^3 + 20p^8q^4 + 4p^7q^5] \]
\[ + [2p^{10}q^2 + 12p^9q^3 + 14p^8q^4 + 10p^7q^5 + 2p^6q^6] \]
\[ = p^{12} + 12p^{11}q + 46p^{10}q^2 + 68p^9q^3 + 36p^8q^4 + 14p^7q^5 + 2p^6q^6 \]

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